Banach spaces. A <u>Banach space</u> is a complete normed space $(V, \|-\|)$.

classical examples: $C_0(X)$, $\mathfrak{M}(X) = C_0(X)^*$,

$\ell_p(X)$, $L_p(X,\mu)$, $1 \leq p \leq \infty$.

Hahn-Banach Theorem. Let $V \subseteq W$ be Banach spaces. Then

$$\begin{tikzcd} W \arrow[dr, dashed, "\exists \tilde{\varphi}"] & \\ V \arrow[u, hook, "\iota"] \arrow[r, "\forall \varphi"'] & \mathbb{C} \end{tikzcd}$$

$\exists \tilde{\varphi}$ ← linear bounded

$\forall \varphi$ ← linear bounded

s.t. $\|\tilde{\varphi}\| = \|\varphi\|$.

Corollary. Every Banach space can be naturally isometrically embedded into $l_\infty(X)$, where $l_\infty(X)$ = all bounded functions on X, $\|f\|_\infty = \sup_{x\in X} |f(x)|$ (One may choose X = the closed unit ball in V^*).

Proof. (for separable Banach spaces) Let (v_n) be a dense sequence in the unit sphere $S(V)$ in the separable Banach space V.

$\forall n \in \mathbb{N}$ use the Hahn-Banach theorem to find a norm-one functional $f_n \in V^*$ s.t. $f_n(v_n) = 1$.

Let $\varphi : V \longrightarrow \mathbb{C}^{\mathbb{N}}$ be defined as

$$\varphi(v) = (f_n(v)).$$

Since $\|\varphi(v)\| = \sup_n |f_n(v)| \leq \sup_n \|f_n\| \cdot \|v\|$ $\varphi(V) \subset \ell_\infty(\mathbb{N})$.

Assume $\|v\| = 1$ and take $\varepsilon \in (0,1)$. Choose $n(\varepsilon)$ so that $\|v_{n(\varepsilon)} - v\| < \varepsilon$. Now, $\forall \varepsilon \in (0,1)$

$$|1 - f_{n(\varepsilon)}(v)| = |f_{n(\varepsilon)}(v_{n(\varepsilon)}) - f_{n(\varepsilon)}(v)| = |f_{n(\varepsilon)}(v_{n(\varepsilon)} - v)|$$

$$\leq \|f_{n(\varepsilon)}\| \cdot \|v_{n(\varepsilon)} - v\| \leq 1 \cdot \varepsilon = \varepsilon$$

$$\Rightarrow \quad \|\varphi(v)\| = \sup_{n \in \mathbb{N}} |f_n(v)| \geq 1.$$

But

$$\forall n \in \mathbb{N} \qquad |f_n(v)| \leq \|f_n\| \cdot \|v\| \leq 1$$

$\Rightarrow \quad \|\varphi(v)\| \leq 1.$

therefore $\quad \|\varphi(v)\| = 1,$ when $\|v\| = 1$

$\Rightarrow \quad \forall v \in V \setminus \{0\} \quad \|\varphi(v)\| = \|v\| \sup_{n \in \mathbb{N}} \left| f_n\left(\frac{v}{\|v\|}\right)\right| = \|v\|.$

$\Rightarrow \quad \varphi$ isometry. $\square$

Classical Theory	Noncommutative Theory
Banach Spaces	Operator Spaces
$\ell_\infty(X)$	$B(H)$
$\ell_\infty(X) \xrightarrow{*} \ell_\infty(X)$	$B(H) \xrightarrow{*} B(H)$
anti-linear involution	anti-linear involution
$f^*(x) = \overline{f(x)}$	$\langle b\psi, \varphi \rangle = \langle \psi, b^*\varphi \rangle$
$(V, \|\cdot\|) \hookrightarrow \ell_\infty(X)$	$(V, ?) \hookrightarrow B(H)$
isometric embedding	?-isometric embedding

? = Matrix Norms. $\forall n \in \mathbb{N}$ the identification

$$B(H^n) = M_n(B(H))$$

defines an operator norm $\|-\|_n$ on $M_n(B(H))$.

Definition. [Arveson] A (concrete) operator space is a norm closed subspace $V \subset B(H)$ together with the matrix norm $\|-\|_n$ on each $M_n(B(H))$.

Examples of operator spaces.

- $L_\infty(X,\mu)$ for some measure space (X,μ)
- $C_0(X)$ or $C_b(X)$, i.e. continuous function algebra vanishing at infinity or bounded on a locally compact topological space X.
- Weak* closed operator algebras of some $B(H)$.
- Every operator algebra, i.e. norm closed subalgebra of some $B(H)$

Weak* topology. Let V be a topological vector space. The weak* topology on its continuous dual V^* is the coarsest topology (i.e. with the fewest open subsets) making all maps $V^* \longrightarrow \mathbb{C}$, $v^* \longmapsto v^*(v)$, $v \in V$ continuous.

(other names: ultraweak or σ-weak topology)

The main property: [Banach-Alaoglu Theorem] The norm one closed ball in the continuous dual V^* of a Banach space V is weak* compact.

In the proof we will use nets (for separable Banach spaces sequences suffice. Separable means that there exists a countable dense subset.)

Definition. [E.H. Moore – H.L. Smith '1922] A net is a function from a directed set (i.e. a nonempty set I with a preorder $\leq$ s.t. $\forall_{i,j\in I}\ \exists_{k\in I}\ i,j \leq k$).

Examples. 1. A sequence, when $I = \mathbb{N}$.

2. A system of Riemann sums, labelled by partitions of the integration interval.

3. A system of compatible sections of a presheaf on the category I of open neighborhoods of a point in a topological space X, when $U \leq V$ if $V \hookrightarrow U$.

Definition. We say that a net $(x_i) : I \to X$ with values in a topological space X converges to $x \in X$, $x_i \longrightarrow x$, iff for every neighborhood U of x there exists $i \in I$ such that for every $j \geq i$ $x_j \in U$.

Examples. 1. Limit of a sequence in a topological space.

2. Riemann integral.

3. The germ of a section of a presheaf at a point.

Fundamental facts about limits of nets.

1. A map $f: X \to Y$ of topological spaces is <u>continuous</u> at $x \in X$ iff for every net $x_i \to x$ we have $f(x_i) \to f(x)$.
2. A subset $Z \subset X$ of a topological space X is <u>closed</u> iff for every net $z_i \to x$ we have $x \in Z$.

Proof. Define

$$\varphi : B_{V^*}(0,1) \longrightarrow \prod_{v \in V} B_{\mathbb{C}}(0, \|v\|),$$
$$v^* \longmapsto (v^*(v))_{v \in V}$$

The right hand side is compact by compactness of all complex discs $B_{\mathbb{C}}(0, \|v\|)$ and the Tychonoff theorem.

It is obvious that φ is one-to-one.

A net $v_i^* \longrightarrow v^*$ in the weak* topology on V^* iff $\varphi(v_i^*) \longrightarrow \varphi(v^*)$ $\Rightarrow$ φ continuous, and so is its inverse $\varphi^{-1} : \varphi(B_{V^*}(0,1)) \longrightarrow B_{V^*}(0,1)$.

Now we show that $\varphi(B_{V^*}(0,1))$ is closed.

$$\varphi(v_i^*) \longrightarrow (z_v)_{v \in V} \in \prod_{v \in V} B_{\mathbb{C}}(0, \|v\|), \qquad \begin{aligned} V &\longrightarrow \mathbb{C} \\ v &\longmapsto z_v \end{aligned}$$

As $\lim_i v_i^*(v) = z_v$ (by definition of weak* topology) one can see that $v \longmapsto z_v$ is a linear functional, which we denote by v^*, s.t. $v^* \in B_{V^*}(0,1)$ and $\varphi(v^*) = (z_v)_{v \in V}$ which means that $\varphi(v_i^*) \longrightarrow \varphi(v^*)$. $\square$

From now on we consider the continuous dual Banach spaces as equipped with the weak* topology.

Definition. $\varphi : V \to W$ a linear map is called <u>completely bounded</u> if

$$\|\varphi\|_{cb} := \sup\{\|M_n(\varphi)\| \mid n \in \mathbb{N}\} < \infty.$$

We denote the space of such maps as $CB(V, W)$.

Theorem. [Arveson-Wittstock-Hahn-Banach Thm]
Let $V \subseteq W \subseteq B(H)$ be operator spaces. Then

$$\begin{tikzcd} W \arrow[dr, dashed, "\exists \tilde{\varphi}"] & \\ V \arrow[u, hook] \arrow[r, "\forall \varphi"'] & B(H). \end{tikzcd}$$

completely bounded

completely bounded

s.t. $\|\tilde{\varphi}\|_{cb} = \|\varphi\|_{cb}$.

Remark. If $\dim H = 1$, $B(H) = \mathbb{C}$, $\|\varphi\|_{cb} = \|\varphi\|$ and we obtain the Hahn-Banach thm.

Proposition. $\varphi : V \longrightarrow C_b(X)$ bounded linear map of Banach spaces is completely bounded with

$$\|\varphi\|_{cb} \leqslant \|\varphi\|.$$

Proof. Given $v_n \in M_n(V)$, $M_n(\varphi)(v_n) \in M_n(C_b(X))$ $= C_b(X, M_n(\mathbb{C}))$ hence

$$\|M_n(\varphi)(v_n)\|_{C_b(X, M_n(\mathbb{C}))} = \sup\{\|M_n(\varphi)(v_n)(x)\|_{M_n(\mathbb{C})} \mid x \in X\}$$

$$= \sup\{|\alpha^* M_n(\varphi)(v)(x)\beta| \mid x \in X, \alpha, \beta \in \mathbb{C}^n, \|\alpha\|_2 = \|\beta\|_2 = 1\}$$

$$= \sup\{|M_n(\varphi)(\alpha^* v \beta)(x)| \mid x \in X, \alpha, \beta \in \mathbb{C}^n, \|\alpha\|_2 = \|\beta\|_2 = 1\}$$

$$\leqslant \|\varphi\| \sup \{ \|\alpha^* v_n \beta\|(x) \mid x \in X, \|\alpha\|_2 = \|\beta\|_2 = 1 \}$$

$$\leqslant \|\varphi\| \cdot \|v_n\|$$

$$\Longrightarrow \sup \{ \|M_n(\varphi)\| \mid n \in \mathbb{N} \} \leq \|\varphi\| . \quad \blacksquare$$

Exercise 5. Show that the transposition map

$$\varphi : M_2(\mathbb{C}) \longrightarrow M_2(\mathbb{C}), \quad v \longmapsto v^T$$

(is completely bounded and) satisfies $\|\varphi\|_{cb} > \|\varphi\|$.

Jun Tomiyama, On the Transpose Map of Matrix Algebras.

<u>Solution</u> Let $v \in M_2(\mathbb{C})$. Then $\|v\| = \|v^T\|$

so $\|\varphi\| = 1$.

$$\left\| M_2(\varphi) \underbrace{\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}}_{X} \right\| = \left\| \underbrace{\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}}_{Y} \right\| = 2 \qquad Y^2 = 2Y,\ Y^T = Y$$

X of norm 1 since unitary $(X^2 = I,\ X^T = X)$

$$\Longrightarrow \|M_2(\varphi)\| \geq 2 \Longrightarrow \|\varphi\|_{cb} \geq 2 > 1 = \|\varphi\|.$$

Haagerup tensor products. $V_1, \dots, V_p$ operator spaces.

Definition. For $u \in M_n(V_1 \otimes \dots \otimes V_p)$ we define the <u>Haagerup norm</u> of u by

$$\|u\|_h := \inf\left\{ \|v_1\| \cdots \|v_p\| \;\middle|\; u = v_1 \underset{M_{n_1}}{\otimes} \cdots \underset{M_{n_{p-1}}}{\otimes} v_p \,,\; v_k \in M_{n_{k-1}, n_k}(V_k) \right\}.$$

$$n_0 = n_p = n$$

These matrix norms determine an operator space structure on $V_1 \otimes \dots \otimes V_p$. The completion in the norm $\|-\|_h$ is denoted $V_1 \overset{h}{\otimes} \dots \overset{h}{\otimes} V_p$ and called the <u>Haagerup tensor product</u>.

Fact. $\overset{h}{\otimes}$ is associative, but in generally not symmetric.

Definition. Given operator spaces $V_1, \ldots, V_p$ and a linear map

$$\varphi : V_1 \otimes \cdots \otimes V_p \longrightarrow W$$

we say that φ is <u>multiplicatively bounded</u> if there is a constant K s.t. for all $n_0 = n_p = n$, $n_1, \ldots, n_{p-1} \in \mathbb{N}$ and $v_k \in M_{n_{k-1}, n_k}(V_k)$,

$$\| M_n(\varphi)(v_1 \otimes_{M_{n_1}} \cdots \otimes_{M_{n_{p-1}}} v_p) \| \leq K \|v_1\| \cdots \|v_p\|.$$

If φ is multiplicatively bounded, the least such K is denoted by $\|\varphi\|_{mb}$ and defines the operator space of such maps

$$CB_m(V_1 \times \cdots \times V_p, W).$$

If the V_k and W are <u>dual operator spaces</u> we say that φ is <u>normal</u> if it is weak* continuous in each variable. The space of such maps we denote by

$$CB_m^\sigma(V_1 \times \cdots \times V_p, W).$$

Definition. A linear map $\varphi: V_1 \otimes \cdots \otimes V_p \to W$ is <u>multiplicatively contractive</u> iff there exists a <u>completely contractive map</u>

$$\tilde{\varphi} : V_1 \overset{h}{\otimes} \cdots \overset{h}{\otimes} V_p \to W$$

extending φ.

Fact. There exists a natural isometry

$$CB_m(V_1 \times \cdots \times V_p, W) \cong CB(V_1 \overset{h}{\otimes} \cdots \overset{h}{\otimes} V_p, W)$$

Proposition. Given complete contractions $\varphi_k : V_k \to W_k$

$\varphi_1 \otimes \cdots \otimes \varphi_p : V_1 \times \cdots \times V_p \to W_1 \overset{h}{\otimes} \cdots \overset{h}{\otimes} W_p$ is multiplicatively contractive, and thus determines a complete contraction $\varphi_1 \overset{h}{\otimes} \cdots \overset{h}{\otimes} \varphi_p : V_1 \overset{h}{\otimes} \cdots \overset{h}{\otimes} V_p \to W_1 \overset{h}{\otimes} \cdots \overset{h}{\otimes} W_p$.

Proof. Proposition 9.2.5 of Effros, Ruan, Operator Spaces, London Math. Soc. Monogr. (N.S.), vol. 23. $\blacksquare$

Definition. We define the <u>extended Haagerup tensor product</u> as

$$V_1 \overset{eh}{\otimes} \cdots \overset{eh}{\otimes} V_p := CB^{\sigma}_m(V_1^* \times \cdots \times V_p^*, \mathbb{C}).$$

and the <u>normal Haagerup tensor product</u> of dual operator spaces

$$V_1^* \overset{\sigma h}{\otimes} \cdots \overset{\sigma h}{\otimes} V_p^* := (V_1 \overset{eh}{\otimes} \cdots \overset{eh}{\otimes} V_p)^*.$$

Facts.

- $CB_m^\sigma(V_1^* \times \cdots \times V_n^*, W^*) = CB^\sigma(V_1^* \overset{\sigma h}{\otimes} \cdots \overset{\sigma h}{\otimes} V_n^*, W^*)$

Corollary. Any weak*-continuous complete contractions $\varphi_k : V_k^* \to W_k^*$ determine a weak*-continuous complete contraction

$$\varphi_1 \overset{\sigma h}{\otimes} \cdots \overset{\sigma h}{\otimes} \varphi_p = (\varphi_{1*} \overset{eh}{\otimes} \cdots \overset{eh}{\otimes} \varphi_{p*})^* :$$

$$V_1^* \overset{\sigma h}{\otimes} \cdots \overset{\sigma h}{\otimes} V_p^* \longrightarrow W_1^* \overset{\sigma h}{\otimes} \cdots \overset{\sigma h}{\otimes} W_p^* .$$

- $\overset{eh}{\otimes}$ is associative, so is $\overset{\sigma h}{\otimes}$ of dual spaces.